Numerical Stability of Hyperbolic Equations in Three Independent Variables

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It is shown that nonsimplicial networks proposed for the solution of hyperbolic equations in three dimensions, which satisfy the Courant-Friedrichs-Lewy condition for stability, should be further analyzed using the von Neumann condition, since the latter is a stronger necessary condition than the former for nonsimplicial networks. Several networks proposed for the solution of three-dimensional supersonic flow fields are evaluated on this basis. The stability predictions for one of the proposed networks are compared with the results of numerical experiments. This method of stability analysis also indicates an approximate "degree" of instability of unstable networks. With the availability of high-speed digital computers, the von Neumann condition can be tested even for complicated difference networks.

Introduction

THE hyperbolic equations in two independent variables that govern two-dimensional steady inviscid supersonic flow fields can be solved numerically using the classical method of characteristics. One obtains the properties at the intersection point of two characteristic lines emanating from base points on the initial value line by the simultaneous solution of two linear difference equations with constant coefficients. The difference domain of dependence of the point being calculated, defined as the convex hull of the base points, coincides with the differential domain of dependence, both being that segment of the initial value line between the base points. Thus, this network satisfies the Courant-Friedrichs-Lewy necessary condition (hereafter denoted by CFL) for numerical stability of linear difference equations with constant coefficients, namely, that the difference domain of dependence must contain the differential domain of dependence.[1] Hahn[2] has shown the CFL condition to be a sufficient condition for stability for all simplicial networks (i.e., networks that determine a new point by using $L + 1$ points of an L-dimensional simplex in the initial value surface of dimension L). Since the network described previously is simplicial ($L = 1$; and a one-dimensional simplex is a line) and satisfies the CFL condition, it is stable.

Whereas the method of characteristics in two independent variables generates only one finite-difference network, an extension of this method to three independent variables can generate a number of finite-difference networks. Fowell[3] discusses several such networks as applied to the numerical solution of three-dimensional steady inviscid supersonic flow fields. In a recent note, Sauerwein and Sussman[4] have evaluated the stability of the various networks discussed by Fowell, using as their sole stability criterion the CFL condition. Such an evaluation is valid for the case of simplicial networks, where the CFL condition is a sufficient condition. However, the unresolved question, as to whether nonsimplicial networks that satisfy the CFL condition are stable, still exists.

Some time ago we initiated a stability study of the various networks discussed by Fowell. A matter of concern to us was the fact that, for three-dimensional, nonsimplicial networks, use of the CFL condition would take into account only the location of those base points that were vertices of the convex hull.‡ As a result, we concentrated our efforts on the study of nonsimplicial networks using the von Neumann condition, which takes into account the location and method of use of all base points. The results of this study[5] show that the von Neumann condition should be used for further analyzing the stability of nonsimplicial networks that satisfy the CFL condition. The purpose of this paper is to report on this study.

Stability Criteria

A necessary condition for stability of linear difference equations with constant coefficients, attributed to von Neumann and described in Ref. 6, is that the eigenvalues of the amplification matrix of the difference equations be less than or equal to one in absolute value. Lax[7] has shown the von Neumann condition to be a sufficient condition for stability for the case of infinitely differentiable data. There appears to be some question as to the sufficiency of the von Neumann condition for arbitrary initial data. Actually, the fact that the von Neumann condition is a necessary and sufficient condition for stability, in the case of infinitely differentiable initial data, is a good demonstration of the strength of its necessity for arbitrary initial data, because physically realizable arbitrary data can be approximated as closely as required by infinitely differentiable data. Richtmyer[8] reported that the von Neumann condition has turned out to be sufficient as well as necessary for stability of all the difference schemes that he had investigated.

Since the CFL condition only takes into consideration the locations of the base points that are vertices of the convex hull, whereas the von Neumann condition takes into account

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‡ For an initial value surface of two dimensions, the convex hull of the base points is the boundary of the union of all triangles formed by joining all pairs of points by straight lines. It follows that each vertex of the convex hull forms an interior angle of less than π rad.

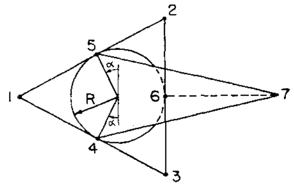

Fig. 1 Simplicial modified tetrahedral characteristic line network.

the location of all base points, as well as the manner in which all base points are used in calculating a new point, it is reasonable to conclude that the von Neumann condition is a stronger necessary condition for stability than the CFL condition for that class of nonsimplicial networks for which the base points are not all vertices of the convex hull. It can furthermore be concluded that, unless different ways of using the same base points are equivalent from the viewpoint of stability, the von Neumann condition is a stronger necessary condition than the CFL condition for networks for which all the base points are vertices of the convex hull. Apparently, such equivalence is the case for simplicial networks (where, by definition, all the base points are vertices of the convex hull) on the basis of Hahn's[2] proof that the CFL condition is a sufficient condition for stability of simplicial networks, despite the fact that it does not take into consideration how one uses the base points. (That one can use the base points of a given network in different ways can be demonstrated by looking at Fig. 1. One can solve for the properties at point 7 by writing appropriate equations along straight lines connecting points 1–3 and point 7; or one can use points 1–3 to define intermediate points 4–6, and then solve for the properties at point 7 by writing appropriate equations along the characteristic lines connecting points 4–6 with point 7.) Apparently, such equivalence is, in general, not the case for that class of nonsimplicial networks wherein all base points are vertices of the convex hull, on the basis of Hahn's proof that the CFL condition is not a sufficient condition for at least one network in this class. Therefore, it is reasonable to conclude that the von Neumann condition is also a stronger necessary condition for stability than the CFL condition for that class of nonsimplicial networks wherein the base points are all vertices of the convex hull.

Since the von Neumann condition is one for the study of linear difference equations with constant coefficients, and since the difference equations governing three-dimensional flow fields are nonlinear, in order to apply the condition to the flow-field equations it is necessary to linearize them. Linearization of the equations governing three-dimensional steady inviscid irrotational supersonic flow yields the two-dimensional wave equation when a suitable transformation of coordinates is used. The authors applied the von Neumann condition to numerical solutions for the derivatives of the dependent variable in the two-dimensional wave equation, using the various proposed three-dimensional difference networks. This method of analyzing the stability of finite-difference schemes for the flow-field equations is heuristic, since it is based on a mathematical study of the stability of linear equations. Of course, these remarks also apply to the use of the CFL condition. A heuristic approach is necessary, since there is no exact test for stability of nonlinear difference equations.

There are a number of sufficient conditions for stability that could be used to further extend the stability analysis. Lax and Richtmyer[6] have shown certain "strengthened" forms of the von Neumann condition to be sufficient conditions for stability of linear equations with constant coefficients. Friedrichs[9] has shown that a sufficient condition for stability of linear symmetric hyperbolic equations with constant coefficients is the positiveness of the coefficient matrices in the difference scheme. Lax[7] extended Friedrichs' condition to nonsymmetric equations.

Difference Networks

As shown by Sauerwein and Sussman,[4] the simplicial tetrahedral characteristic line network proposed by Fowell[3] is unstable, since it violates the CFL condition. Sauerwein and Sussman[4] have proposed a variation of this network (Fig. 1) that they term the modified tetrahedral characteristic line network. The properties at the base points 4–6 are obtained by interpolation between points 1–3, where points 1–3 are located by the unit mesh calculations of the previous calculation level, no interpolation being required. The points 4–6 are located at the points where the inscribed circle of the triangle 1–3 touches the sides of the triangle. As shown by Sauerwein and Sussman, this network is stable, since it is simplicial (points 1–3 determine a triangle, which is a two-dimensional simplex), and since it satisfies the CFL condition. The authors applied the von Neumann test for stability to a numerical solution of the two-dimensional wave equation using this stable network.

The von Neumann condition requires that the eigenvalues of the amplification matrix be less than or equal to one in absolute value for all possible combinations of Fourier indices n, m occurring in a Fourier series solution of the difference equations. Of course, such a complete test is impossible when performed numerically. An eigenvalue index i is defined which encompasses all sets of indices n, m for which $|n| \leq i$, $|m| \leq i$, and one or both of n and $m = \pm i$. It is

Table 1 Max $|\lambda|$ of the amplification matrix for the nonsimplicial form of the modified tetrahedral characteristic line network

	α, deg					
i	10	30	40	50	60	80
0	1.00	1.00	1.00	1.00	1.00	1.00
1	1.01	1.00	1.00	1.01	1.01	1.00
1	1.02	1.01	1.03	1.03	1.03	1.05
3	1.02	1.01	1.06	1.05	1.01	1.04
4	1.02	1.00	1.08	1.08	1.02	0.978
5	1.02	0.999	1.10	1.09	1.02	0.985
6	1.00	0.989	1.10	1.10	1.02	0.931
7	0.950	0.981	1.08	1.09	1.02	0.926
8	0.918	0.973	1.06	1.06	1.00	0.906
9	0.887	0.966	1.01	1.02	0.975	0.843
10	0.853	0.958	0.993	0.980	0.960	0.844
11	0.829	0.951	0.990	0.977	0.954	0.816
12	0.807	0.944	0.988	0.981	0.949	0.757
13	0.779	0.939	0.988	1.04	0.940	0.712
14	0.760	0.935	1.02	1.09	0.931	0.696
15	0.734	0.934	1.04	1.10	0.929	0.669

Table 2 Max $|\lambda|$ of the amplification matrix for the pentahedral characteristic surface network

	a/b					
i	1	$\frac{5}{6}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$
0	1.00	1.00	1.00	1.00	1.00	1.00
1	0.996	0.995	0.994	0.992	0.983	0.972
2	0.999	0.999	0.999	0.998	0.997	0.995
3	0.998	0.998	0.997	0.996	0.992	0.987
4	0.999	0.997	0.996	0.995	0.990	0.983
5	0.999	0.999	0.999	0.999	0.998	0.997
6	0.995	0.994	0.992	0.989	0.978	0.972
7	0.999	0.999	0.999	0.999	0.999	0.999
8	0.998	0.996	0.995	0.992	0.993	0.975
9	0.999	0.999	0.998	0.998	0.996	0.993
10	0.998	0.998	0.998	0.997	0.994	0.991
11	0.999	0.996	0.995	0.993	0.987	0.991
12	0.999	0.999	0.999	0.999	0.999	0.999
13	0.999	0.994	0.992	0.988	0.976	0.959
14	0.999	0.999	0.999	0.999	0.999	0.998
15	0.999	0.997	0.996	0.999	0.999	0.999

reasonable to assume that calculations up to $i = 15$ take into account an adequate number of terms in the Fourier solution. For a given value of i, "max $|\lambda|$" is defined as the absolute value of that eigenvalue which has the greatest absolute value of all eigenvalues calculated for all n, m combinations for that value of i.

Eigenvalue calculations up to $i = 15$ for the preceding simplicial network yielded values of "max $|\lambda|$" that were either 1.00 or 0.999 for all values of α (see Fig. 1), three digits being the limit of significance in these calculations. The details of these calculations are presented in Ref. 5. These results are in accord with the proved stability of this network. The fact that all "max $|\lambda|$" are very near unity leads one to think that it may be possible to show exactly that they are unity, but we have been unable to do so.

A nonsimplicial variation of the modified tetrahedral characteristic line network is one in which the properties at points 4–6 (Fig. 1) are not obtained by interpolation between points 1–3. In this network, the properties at all points 1–6 are obtained by interpolation between mesh points calculated in the previous calculation level (Fig. 2). In the wave equation analysis, the properties at points 1–6 were obtained by interpolation between a given square mesh of points with spacing equal to R, one of which is the center of the initial value circle. In Fig. 2 interpolation points are shown by circles, and the convex hull is shown by long-dashed lines. This network satisfies the CFL condition for stability. The eigenvalues of the amplification matrix for this network are given in Table 1, which indicates that this network is unstable. The "degree" of instability is small, as evidenced by eigenvalues not appreciably greater than 1.0 in absolute value. The fact that this network satisfies the CFL condition, and yet violates the von Neumann condition, confirms the authors' conclusion that the von Neumann condition is a stronger necessary condition than the CFL condition for nonsimplicial networks.

A second nonsimplicial network is the pentahedral characteristic surface network proposed by Brong[10] for the solution of the three-dimensional flow problem. The network is shown in Fig. 3. Points 7 and 8 are determined by the intersection of the characteristic surfaces containing the sides of the quadrilateral 1234. For the purposes of the stability analysis, 1234 was taken to be rectangular. The properties at point 7 are determined by difference equations along 17, 27, and the characteristic 57. Point 8 is calculated similarly. For the purpose of constructing a practical over-all mesh, the properties at points 7 and 8 are interpolated to yield the properties at a suitable point along line 78. This point will then serve as a base point in a unit mesh calculation in the next calculation level. The properties at the base points 1–4 were also obtained by the same type of interpolation. A desirable feature of this unit mesh is that it is especially suitable for the construction of a practical over-all mesh. This unit mesh satisfies the CFL condition. The eigenvalues of the amplification matrix for this network as applied to the wave equation are given in Table 2, which indicates that the von Neumann condition is also satisfied. Brong has found no instabilities in numerical solutions of three-dimensional flow fields using this network.

A final example of the class of nonsimplicial networks, wherein the base points in the initial surface are not all vertices of the convex hull of the network, is a nonsimplicial version of the characteristic line network. This network and numerical experiments using it are described in the next section.

Holt,[11] Ferrari,[12] and Butler[13] have proposed nonsimplicial networks for the solution of the three-dimensional flow problem. As already reported by Sauerwein and Sussman, these networks all violate the CFL necessary condition for stability, if the base points in any unit mesh calculation are points located by the unit mesh calculations of the previous calculation level. Hence, further stability analysis is not necessary for these cases. However, the introduction of interpolation can sufficiently increase the size of the difference domain of dependence, so as to satisfy the CFL condition. It was noted by Sauerwein and Sussman that apparently this is what occurred in numerical solutions of the flow problem performed by Moretti,[14] using the basic Ferrari network, and by Butler,[13] where there were no evidences of instability. It is important to stress here that the fact that interpolation causes the CFL condition to be satisfied does not necessarily mean these nonsimplicial networks are stable, since the CFL condition is only a necessary condition and a weak one at

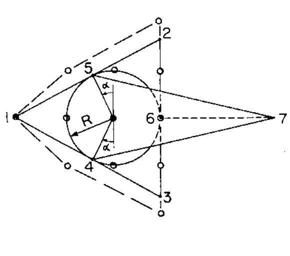

Fig. 2 Nonsimplicial modified tetrahedral characteristic line network.

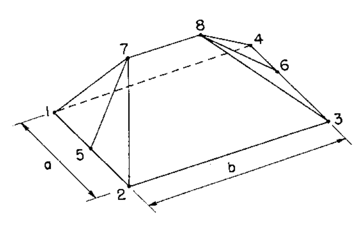

Fig. 3 Pentahedral characteristic surface network.

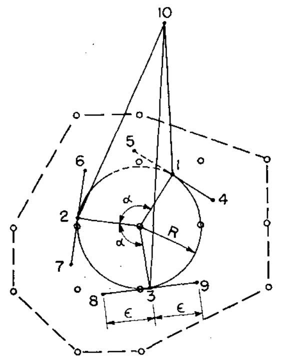

Fig. 4 Nonsimplicial form of the tetrahedral characteristic line network.

that. It is possible that instabilities were not encountered in the solutions reported previously, only because an insufficient number of levels were calculated. Our method of analysis, using the von Neumann condition applied to a linearized form of the governing equations, can be used to test further these networks for instability, provided the specification of the unit meshes takes into account all required interpolations. Because of lack of knowledge as to the exact methods of interpolation used by Moretti and Butler, the authors have not carried out these analyses. If this information were available, it would be straightforward to make these analyses. The authors have devised a variation of the Holt network, with no interpolation required, which, when used to solve the two-dimensional wave equation, satisfies the von Neumann condition.[5]

Numerical Experiments

The authors have carried out numerical solutions of the two-dimensional wave equation for the purpose of verifying the results of the von Neumann test for stability. As shown by Sauerwein and Sussman, the simplicial tetrahedral characteristic line network proposed by Fowell[3] is unstable, since it violates the CFL condition. The authors have carried out numerical solutions using a nonsimplicial variation of this network[5] (Fig. 4). The derivatives at the base points 1–3 are obtained by differencing the properties at points 4–9, each of which is a distance ϵ from one of the points 1–3, as shown in Fig. 3. The properties at points 4–9 are obtained by two-way linear interpolation between a square mesh of points with spacing R, one of which is the center of the initial value circle. Again, the convex hull is indicated by long-dashed lines. This network was used to solve for the derivatives P_i (i = 1–3) of the dependent variable in the two-dimensional wave equation. The eigenvalues of the amplification matrix for this numerical solution are given in Fig. 5. The results of this numerical solution, using linear initial data, are given in Fig. 6. N is the number of calculation levels. A "P_i stability number" is defined as the average over all the unit mesh calculations at the given calculation level of the absolute value of the error in the numerical solution, that is $|[P_i\text{ (numerical)} - P_i\text{ (exact)}]/P_i\text{(exact)}|$. P_i (exact) comes from the known analytical solution of the two-dimensional wave equation. The stability numbers for each of the three P_i derivatives were averaged at each calculation level to yield the "average stability number" plotted in Fig. 6. The occurrence of instability is evidenced in Fig. 6 by an "exponentially" increasing stability number. Note that the results of Fig. 5 predict fairly accurately the order of occurrence of instability of the various cases in Fig. 6. The four cases that did not exhibit instability in Fig. 6 would, on the basis of Fig. 5, eventually become unstable if a sufficiently large number of levels were calculated. Thus, the von Neumann condition not only can determine that a given numerical solution of linear difference equations is unstable, but it can predict the approximate "degree" of instability.

It can be shown geometrically that the $\alpha = 120°$; $R/\epsilon = 5$ and $\alpha = 60°$; $R/\epsilon = 2, 5$ cases all violate the CFL condition, and thus, one would expect the result that the von Neumann condition is also violated for these cases. On the other hand, it can be shown that the cases $\alpha = 120°$; $R/\epsilon = \frac{1}{2}$, 1, 2 and $\alpha = 60°$; $R/\epsilon = \frac{1}{2}$, 1 all satisfy the CFL condition; yet the von Neumann condition reveals these cases to be unstable.

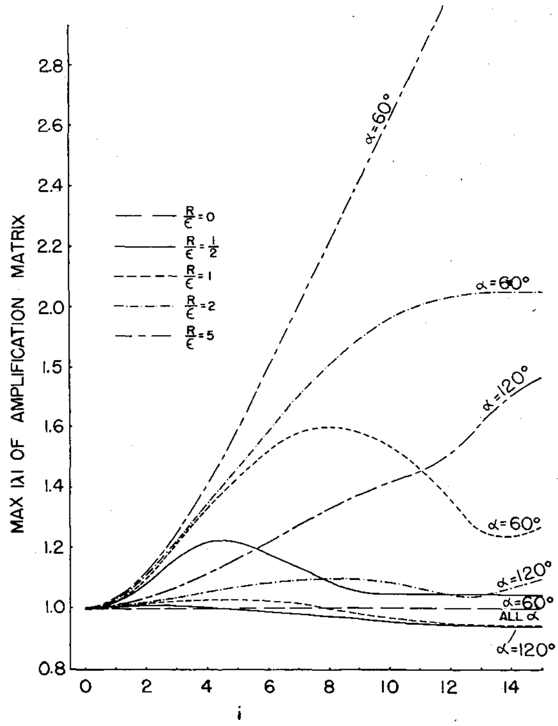

Fig. 5 Max $|\lambda|$ of the amplification matrix for the nonsimplicial form of the tetrahedral characteristic line network.

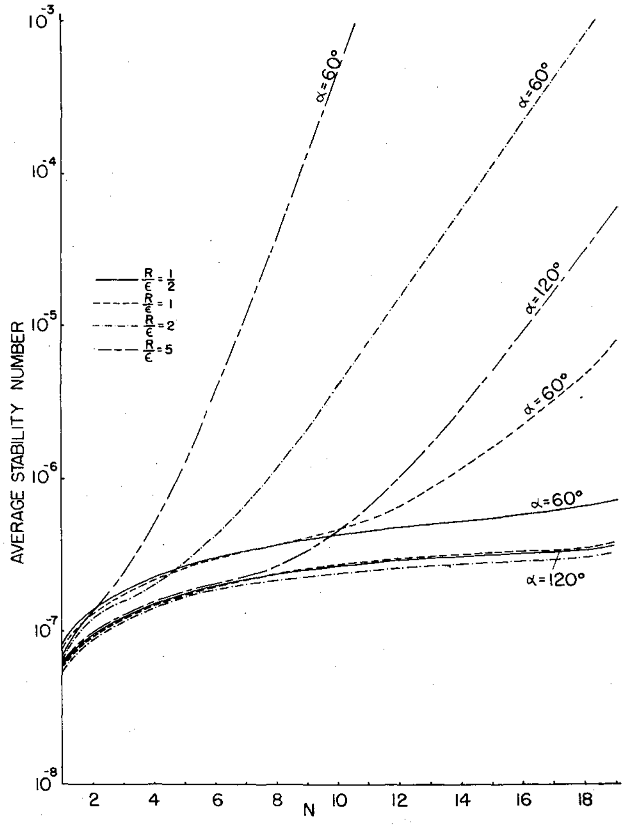

Fig. 6 Numerical solution of two-dimensional wave equation using the nonsimplicial form of tetrahedral characteristic line network.

This is another case where the von Neumann condition is violated even though the CFL condition is satisfied.

Conclusions

When numerically solving hyperbolic equations, the CFL condition is a necessary condition for stability but is a sufficient condition only for the case of simplicial networks. In the case of nonsimplicial networks it is, in fact, a weaker necessary condition than the von Neumann condition. As a result, nonsimplicial networks that satisfy the CFL condition should be further tested for instability using the von Neumann condition applied to a linearized form of the governing difference equations. This method will indicate an approximate "degree" of instability for networks found to be unstable. The availability of high-speed digital computers makes the use of the von Neumann condition feasible no matter how complex the difference network.

Nonsimplicial networks that satisfy both the CFL and the von Neumann conditions, as applied to the linearized equations, can be presumed to be stable, keeping in mind the question as to the sufficiency of the von Neumann condition for arbitrary initial data.

Application of the von Neumann condition to a linearized form of the three-dimensional steady flow equations, using various proposed networks, yields the following results. The modified tetrahedral characteristic line network, which is known to be stable, satisfies the von Neumann condition. A nonsimplicial form of the modified tetrahedral characteristic line network is found to be unstable, despite the fact that it satisfies the CFL condition. The nonsimplicial pentahedral characteristic surface network and the authors' variation of the Holt network satisfy both the CFL and the von Neumann conditions for stability. A nonsimplicial version of the tetrahedral characteristic line network was shown to be unstable by the von Neumann condition, even though it satisfied the CFL condition in some cases.

References

[1] Courant, R., Friedrichs, K. O., and Lewy, H., "Uber die Partiellen Differenzialgleichungen der Mathematischen Physik," Math. Ann. **100**, 32–74 (1928).

[2] Hahn, S. G., "Stability criteria for difference schemes," Commun. Pure Appl. Math. **11**, 243–255 (1958).

[3] Fowell, L. R., "Flow field analysis for lifting re-entry configurations by the method of characteristics," IAS Preprint 61-208-1902 (June 1961).

[4] Sauerwein, H. and Sussman, M., "Numerical stability of the three-dimensional method of characteristics," AIAA J. **2**, 387–389 (1964).

[5] Heie, H. and Leigh, D. C. "Stability of the numerical solution of hyperbolic partial differential equations in three independent variables using the method of characteristics," Dept. of Aerospace and Mechanical Sciences Rept. 705, Princeton Univ. (October 1964).

[6] Lax, P. D. and Richtmyer, R. D., "Survey of the stability of linear finite difference equations," Commun. Pure Appl. Math. **9**, 267–293 (1956).

[7] Lax, P. D., "Differential equations, difference equations and matrix theory," Commun. Pure Appl. Math. **11**, 175–194 (1958).

[8] Richtmyer, R. D., *Difference Methods for Initial-Value Problems* (Interscience Publishers, Inc., New York, 1957), 1st ed., Chap. IV, p. 71.

[9] Friedrichs, K. O., "Symmetric hyperbolic linear differential equations," Commun. Pure Appl. Math. **7**, 345–392 (1954).

[10] Brong, E. A., "A method of characteristics for solution of three-dimensional supersonic shock layer flows," General Electric TIS 64SD271 (to be published).

[11] Holt, M., "The method of characteristics for steady supersonic rotational flow in three dimensions," J. Fluid Mech. **1**, 409–423 (1956).

[12] Ferrari, C., "Interference between wing and body at supersonic speeds-analysis by the method of characteristics," J. Aeronaut. Sci. **16**, 411–434 (1949).

[13] Butler, D. S., "The numerical solution of hyperbolic systems of partial differential equations in three independent variables," Proc. Roy. Soc. (London) **255A**, 232–252 (1960).

[14] Moretti G., "Three-dimensional supersonic flow computations," AIAA J. 1, 2192–2193 (1963).